Automata and Formal Languages (7)

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CFGs

A context-free grammar (CFG) G consists of:

- a finite set of nonterminal symbols (upper case)
- a finite terminal symbols or tokens (lower case)
- a start symbol (which must be a nonterminal)
- a set of rules

$$A \rightarrow \text{rhs}_1 | \text{rhs}_2 | \dots$$

where rhs are sequences involving terminals and nonterminals (can also be empty).

CFGs

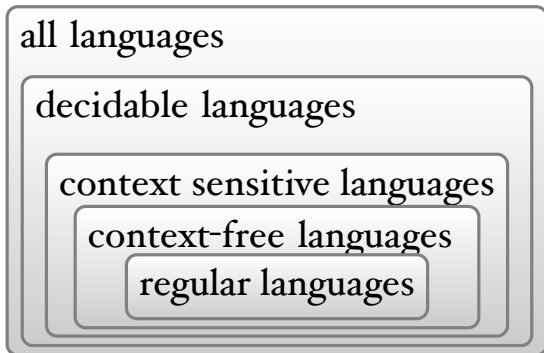A context-free grammar (CFG) G consists of:

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Hierarchy of Languages

Recall that languages are sets of strings.

Arithmetic Expressions

A grammar for arithmetic expressions and numbers:

$$E \rightarrow E \cdot + \cdot E \mid E \cdot * \cdot E \mid (\cdot E \cdot) \mid N$$
$$N \rightarrow N \cdot N \mid 0 \mid 1 \mid \ldots \mid 9$$

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Numbers

$$N \rightarrow N \cdot N \mid 0 \mid 1 \mid \ldots \mid 9$$

A non-left-recursive, non-ambiguous grammar for numbers:

$$N \rightarrow 0 \cdot N \mid 1 \cdot N \mid \ldots \mid 0 \mid 1 \mid \ldots \mid 9$$

Operator Precedences

To disambiguate

$$E \quad \to \quad E \cdot + \cdot E \mid E \cdot * \cdot E \mid (\cdot E \cdot) \mid N$$

Decide on how many precedence levels, say

highest for $()$, medium for $*$, lowest for $+$

$$
\begin{aligned}
E_{low} &\to E_{med} \cdot + \cdot E_{low} \mid E_{med} \\
E_{med} &\to E_{hi} \cdot * \cdot E_{med} \mid E_{hi} \\
E_{hi} &\to (\cdot E_{low} \cdot) \mid N
\end{aligned}
$$

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What happens with $1 + 3 + 4$?

Removing Left-Recursion

The rule for numbers is directly left-recursive:

$$N \;\to\; N \cdot N \mid 0 \mid 1 \quad (\ldots)$$

Translate

$$
\begin{aligned}
N \;\to\;& N \cdot \alpha \\
\mid\;& \beta
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
N \;\to\;& \beta \cdot N' \\
N' \;\to\;& \alpha \cdot N' \\
\mid\;& \epsilon
\end{aligned}
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Which means

$$
\begin{aligned}
N \;\rightarrow\;& 0 \cdot N' \mid 1 \cdot N' \\
N' \;\rightarrow\;& N \cdot N' \mid \epsilon
\end{aligned}
$$

Chomsky Normal Form

All rules must be of the form

$$A \rightarrow a$$

or

$$A \rightarrow B \cdot C$$

No rule can contain ϵ.

ϵ-Removal

1. If $A \rightarrow \alpha \cdot B \cdot \beta$ and $B \rightarrow \epsilon$ are in the grammar, then add $A \rightarrow \alpha \cdot \beta$ (iterate if necessary).
2. Throw out all $B \rightarrow \epsilon$.

$$N \rightarrow 0 \cdot N' \mid 1 \cdot N'$$
$$N' \rightarrow N \cdot N' \mid \epsilon$$

$$N \rightarrow 0 \cdot N' \mid 1 \cdot N' \mid 0 \mid 1$$
$$N' \rightarrow N \cdot N' \mid N \mid \epsilon$$

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$$N \rightarrow \text{o} \cdot N' \mid \text{1} \cdot N'$$
$$N' \rightarrow N \cdot N' \mid \epsilon$$

$$N \rightarrow \text{o} \cdot N' \mid \text{1} \cdot N' \mid \text{o} \mid \text{1}$$
$$N' \rightarrow N \cdot N' \mid N \mid \epsilon$$

$$N \rightarrow \text{o} \cdot N' \mid \text{1} \cdot N' \mid \text{o} \mid \text{1}$$
$$N' \rightarrow N \cdot N' \mid N$$

$$N \rightarrow \text{o} \cdot N \mid \text{1} \cdot N \mid \text{o} \mid \text{1}$$

CYK Algorithm

If grammar is in Chomsky normalform ...

$$S \;\rightarrow\; N \cdot P$$
$$P \;\rightarrow\; V \cdot N$$
$$N \;\rightarrow\; N \cdot N$$
$$N \;\rightarrow\; \text{students} \mid \text{Jeff} \mid \text{geometry} \mid \text{trains}$$
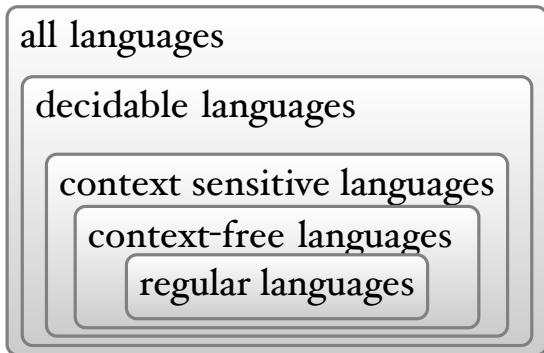$$V \;\rightarrow\; \text{trains}$$

Jeff trains geometry students

CYK Algorithm

- fastest possible algorithm for recognition problem
- runtime is $O(n^3)$

- grammars need to be transferred into CNF

Hierarchy of Languages

Recall that languages are sets of strings.

all languages
decidable languages
context sensitive languages
context-free languages
regular languages

Context Sensitive Grms

$$S \Rightarrow bSAA \mid \epsilon$$
$$A \Rightarrow a$$
$$bA \Rightarrow Ab$$

Context Sensitive Grms

$$S \Rightarrow bSAA \mid \epsilon$$
$$A \Rightarrow a$$
$$bA \Rightarrow Ab$$

$$S \Rightarrow \ldots \Rightarrow^? \textit{"ababaa"}$$

$$
\begin{array}{rcl}
\textit{Stmt} & \rightarrow & \texttt{skip} \\
& | & \textit{Id} \; \texttt{:=} \; \textit{AExp} \\
& | & \texttt{if} \; \textit{BExp} \; \texttt{then} \; \textit{Block} \; \texttt{else} \; \textit{Block} \\
& | & \texttt{while} \; \textit{BExp} \; \texttt{do} \; \textit{Block} \\
& | & \texttt{read} \; \textit{Id} \\
& | & \texttt{write} \; \textit{Id} \\
& | & \texttt{write} \; \textit{String} \\[1em]
\textit{Stmts} & \rightarrow & \textit{Stmt} \; \texttt{;} \; \textit{Stmts} \\
& | & \textit{Stmt} \\[1em]
\textit{Block} & \rightarrow & \texttt{\{} \; \textit{Stmts} \; \texttt{\}} \\
& | & \textit{Stmt} \\[1em]
\textit{AExp} & \rightarrow & ... \\
\textit{BExp} & \rightarrow & ...
\end{array}
$$

```
1  write "Fib";
2  read n;
3  minus1 := 0;
4  minus2 := 1;
5  while n > 0 do {
6          temp := minus2;
7          minus2 := minus1 + minus2;
8          minus1 := temp;
9          n := n - 1
10 };
11 write "Result";
12 write minus2
```

An Interpreter

$$
\begin{array}{l}
\{ \\
\quad x := 5; \\
\quad y := x * 3; \\
\quad y := x * 4; \\
\quad x := u * 3 \\
\}
\end{array}
$$

- the interpreter has to record the value of x before assigning a value to y

An Interpreter

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$$
$$
\}
$$

- the interpreter has to record the value of x before assigning a value to y
- eval(stmt, env)

Interpreter

$$\text{eval}(n, E) \stackrel{\text{def}}{=} n$$

$$\text{eval}(x, E) \stackrel{\text{def}}{=} E(x) \quad \text{lookup } x \text{ in } E$$

$$\text{eval}(a_1 + a_2, E) \stackrel{\text{def}}{=} \text{eval}(a_1, E) + \text{eval}(a_2, E)$$

$$\text{eval}(a_1 - a_2, E) \stackrel{\text{def}}{=} \text{eval}(a_1, E) - \text{eval}(a_2, E)$$

$$\text{eval}(a_1 * a_2, E) \stackrel{\text{def}}{=} \text{eval}(a_1, E) * \text{eval}(a_2, E)$$

$$\text{eval}(a_1 = a_2, E) \stackrel{\text{def}}{=} \text{eval}(a_1, E) = \text{eval}(a_2, E)$$

$$\text{eval}(a_1 \mathbin{!=} a_2, E) \stackrel{\text{def}}{=} \neg(\text{eval}(a_1, E) = \text{eval}(a_2, E))$$

$$\text{eval}(a_1 < a_2, E) \stackrel{\text{def}}{=} \text{eval}(a_1, E) < \text{eval}(a_2, E)$$

Interpreter (2)

$$\text{eval}(\text{skip}, E) \quad \overset{\text{def}}{=} \quad E$$

$$\text{eval}(x := a, E) \quad \overset{\text{def}}{=} \quad E(x \mapsto \text{eval}(a, E))$$

$$\text{eval}(\text{if } b \text{ then } cs_1 \text{ else } cs_2, E) \overset{\text{def}}{=}$$
$$\qquad \text{if } \text{eval}(b, E) \text{ then } \text{eval}(cs_1, E)$$
$$\qquad\qquad \text{else } \text{eval}(cs_2, E)$$

$$\text{eval}(\text{while } b \text{ do } cs, E) \overset{\text{def}}{=}$$
$$\qquad \text{if } \text{eval}(b, E)$$
$$\qquad \text{then } \text{eval}(\text{while } b \text{ do } cs, \text{eval}(cs, E))$$
$$\qquad \text{else } E$$

$$\text{eval}(\text{write } x, E) \quad \overset{\text{def}}{=} \quad \{ \text{println}(E(x)) \; ; \; E \}$$

Test Program

```
1   start := 1000;
2   x := start;
3   y := start;
4   z := start;
5   while 0 < x do {
6     while 0 < y do {
7       while 0 < z do { z := z - 1 };
8       z := start;
9       y := y - 1
10    };
11    y := start;
12    x := x - 1
13  }
```

Interpreted Code
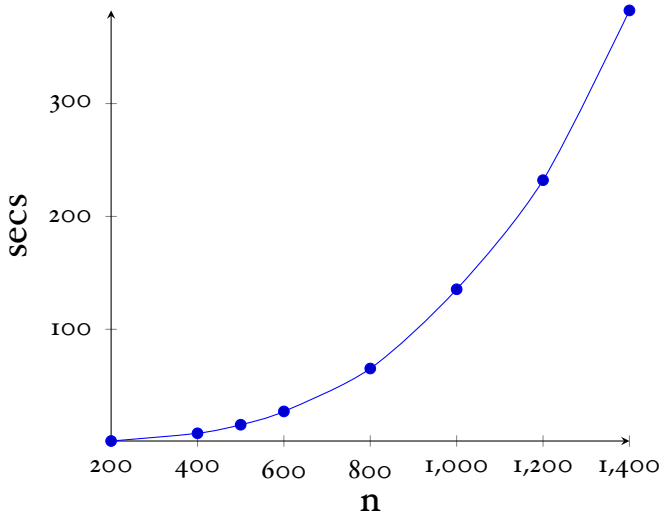
Java Virtual Machine

- introduced in 1995
- is a stack-based VM (like Postscript, CLR of .Net)
- contains a JIT compiler
- many languages take advantage of JVM's infrastructure (JRE)
- is garbage collected $\Rightarrow$ no buffer overflows
- some languages compile to the JVM: Scala, Clojure...