# Automata and Formal Languages (2)


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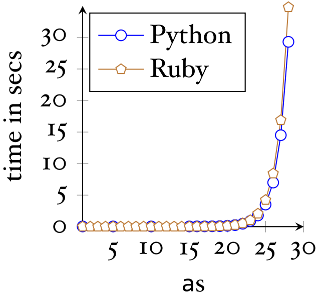
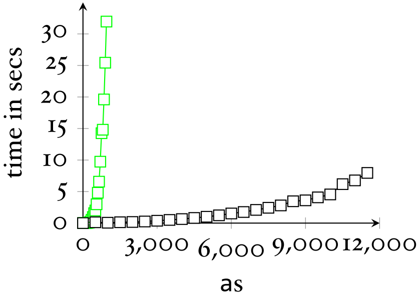
# An Efficient Regular Expression Matcher



# Languages

- A **language** is a set of strings, for example

$$\{[], hello, foobar, a, abc\}$$

- **Concatenation** of strings and languages

$$foo \mathbin{@} bar \quad = \quad foobar$$

$$A \mathbin{@} B \quad \stackrel{\text{def}}{=} \quad \{s_1 \mathbin{@} s_2 \mid s_1 \in A \wedge s_2 \in B\}$$

For example $A = \{foo, bar\}$, $B = \{a, b\}$

$$A \mathbin{@} B = \{fooa, foob, bara, barb\}$$

# The Power Operation

- The **Power** of a language:

$$A^0 \stackrel{\text{def}}{=} \{[]\}$$
$$A^{n+1} \stackrel{\text{def}}{=} A @ A^n$$

For example

$$A^4 = A @ A @ A @ A$$
$$A^0 \stackrel{\text{def}}{=} \{[]\}$$

# Homework Question

- Say $A = \{[a], [b], [c], [d]\}$.

How many strings are in $A^4$?

# Homework Question

- Say $A = \{[a], [b], [c], [d]\}$.

How many strings are in $A^4$?

What if $A = \{[a], [b], [c], []\}$;
how many strings are then in $A^4$?

# The Star Operation

- The **Star** of a language:

$$A^* \stackrel{\text{def}}{=} \bigcup_{0 \leq n} A^n$$

This expands to

$$A^0 \cup A^1 \cup A^2 \cup A^3 \cup A^4 \cup \ldots$$

$$\{[]\} \;\cup\; A \;\cup\; A @ A \;\cup\; A @ A @ A \;\cup\; A @ A @ A @ A \cup \ldots$$

# Semantic Derivative

- The **Semantic Derivative** of a language wrt to a character $c$:

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

For $A = \{foo, bar, frak\}$ then

$$\begin{aligned} Der\,f\,A &= \{oo, rak\} \\ Der\,b\,A &= \{ar\} \\ Der\,a\,A &= \varnothing \end{aligned}$$

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We can extend this definition to strings

$$Ders\,s\,A = \{s' \mid s@s' \in A\}$$

# Regular Expressions

Their inductive definition:

| | | | |
|---|---|---|---|
| $r$ | $::=$ | $\varnothing$ | null |
| | $\mid$ | $\epsilon$ | empty string / "" / [] |
| | $\mid$ | $c$ | character |
| | $\mid$ | $r_1 \cdot r_2$ | sequence |
| | $\mid$ | $r_1 + r_2$ | alternative / choice |
| | $\mid$ | $r^*$ | star (zero or more) |

Th

```
abstract class Rexp
case object NULL extends Rexp
case object EMPTY extends Rexp
case class CHAR(c: Char) extends Rexp
case class ALT(r1: Rexp, r2: Rexp) extends Rexp
case class SEQ(r1: Rexp, r2: Rexp) extends Rexp
case class STAR(r: Rexp) extends Rexp
```

| | | |
|---|---|---|
| $r$ ::= | $\varnothing$ | null |
| \| | $\epsilon$ | empty string / "" / [] |
| \| | $c$ | character |
| \| | $r_1 \cdot r_2$ | sequence |
| \| | $r_1 + r_2$ | alternative / choice |
| \| | $r^*$ | star (zero or more) |

# The Meaning of a Regular Expression

$$
\begin{array}{rcl}
L(\varnothing) & \stackrel{\text{def}}{=} & \varnothing \\
L(\epsilon) & \stackrel{\text{def}}{=} & \{[]\} \\
L(c) & \stackrel{\text{def}}{=} & \{[c]\} \\
L(r_1 + r_2) & \stackrel{\text{def}}{=} & L(r_1) \cup L(r_2) \\
L(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & L(r_1) \,@\, L(r_2) \\
L(r^*) & \stackrel{\text{def}}{=} & (L(r))^*
\end{array}
$$

$L$ is a function from regular expressions to sets of strings

$L : \text{Rexp} \Rightarrow \text{Set}[\text{String}]$

What is $L(a^*)$?

# When Are Two Regular Expressions Equivalent?

$$r_1 \equiv r_2 \quad \stackrel{\text{def}}{=} \quad L(r_1) = L(r_2)$$

# Concrete Equivalences

$$
\begin{array}{rcl}
(a + b) + c & \equiv & a + (b + c) \\
a + a & \equiv & a \\
a + b & \equiv & b + a \\
(a \cdot b) \cdot c & \equiv & a \cdot (b \cdot c) \\
c \cdot (a + b) & \equiv & (c \cdot a) + (c \cdot b)
\end{array}
$$

# Concrete Equivalences

$$
\begin{array}{rcl}
(a+b)+c & \equiv & a+(b+c) \\
a+a & \equiv & a \\
a+b & \equiv & b+a \\
(a\cdot b)\cdot c & \equiv & a\cdot(b\cdot c) \\
c\cdot(a+b) & \equiv & (c\cdot a)+(c\cdot b)
\end{array}
$$

$$
\begin{array}{rcl}
a\cdot a & \not\equiv & a \\
a+(b\cdot c) & \not\equiv & (a+b)\cdot(a+c)
\end{array}
$$

# Corner Cases

$$
\begin{aligned}
a \cdot \varnothing &\not\equiv a \\
a + \epsilon &\not\equiv a \\
\epsilon &\equiv \varnothing^* \\
\epsilon^* &\equiv \epsilon \\
\varnothing^* &\not\equiv \varnothing
\end{aligned}
$$

# Simplification Rules

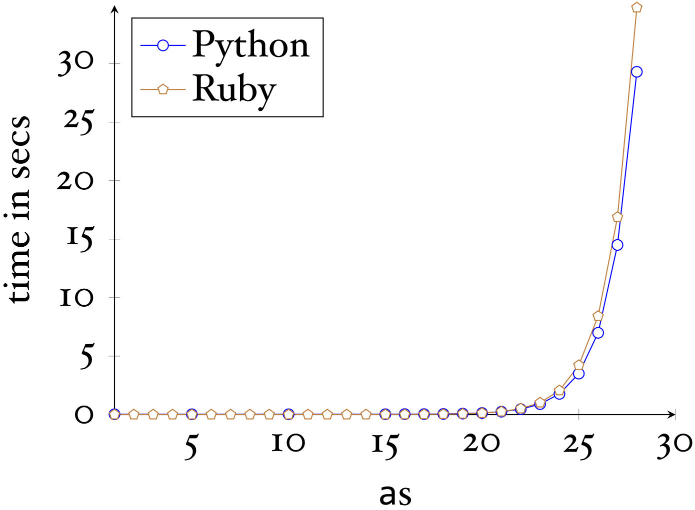
$$
\begin{aligned}
r + \varnothing &\equiv r \\
\varnothing + r &\equiv r \\
r \cdot \epsilon &\equiv r \\
\epsilon \cdot r &\equiv r \\
r \cdot \varnothing &\equiv \varnothing \\
\varnothing \cdot r &\equiv \varnothing \\
r + r &\equiv r
\end{aligned}
$$

# The Specification for Matching

A regular expression $r$ matches a string $s$ if and only if

$$s \in L(r)$$

$$(a^?\{n\}) \cdot a\{n\}$$



# Evil Regular Expressions

- Regular expression Denial of Service (ReDoS)
- Evil regular expressions
  - $(a^{?\{n\}}) \cdot a^{\{n\}}$
  - $(a^+)^+$
  - $([a\text{-}z]^+)^*$
  - $(a + a \cdot a)^+$
  - $(a + a?)^+$

# A Matching Algorithm

...whether a regular expression can match the empty string:

$$
\begin{array}{lcl}
\textit{nullable}(\varnothing) & \stackrel{\text{def}}{=} & \textit{false} \\
\textit{nullable}(\epsilon) & \stackrel{\text{def}}{=} & \textit{true} \\
\textit{nullable}(c) & \stackrel{\text{def}}{=} & \textit{false} \\
\textit{nullable}(r_1 + r_2) & \stackrel{\text{def}}{=} & \textit{nullable}(r_1) \vee \textit{nullable}(r_2) \\
\textit{nullable}(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \textit{nullable}(r_1) \wedge \textit{nullable}(r_2) \\
\textit{nullable}(r^*) & \stackrel{\text{def}}{=} & \textit{true}
\end{array}
$$

# The Derivative of a Rexp

If $r$ matches the string $c::s$, what is a regular expression that matches just $s$?

$der\,c\,r$ gives the answer, Brzozowski 1964

# The Derivative of a Rexp

$$
\begin{array}{lcl}
der\,c\,(\varnothing) & \stackrel{\text{def}}{=} & \varnothing \\
der\,c\,(\epsilon) & \stackrel{\text{def}}{=} & \varnothing \\
der\,c\,(d) & \stackrel{\text{def}}{=} & \text{if } c = d \text{ then } \epsilon \text{ else } \varnothing \\
der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2 \\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if } nullable(r_1) \\
 & & \text{then } (der\,c\,r_1) \cdot r_2 + der\,c\,r_2 \\
 & & \text{else } (der\,c\,r_1) \cdot r_2 \\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r) \cdot (r^*)
\end{array}
$$

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der\,c\,(r_1 + r_2) & \stackrel{\text{def}}{=} & der\,c\,r_1 + der\,c\,r_2\\
der\,c\,(r_1 \cdot r_2) & \stackrel{\text{def}}{=} & \text{if}\; nullable(r_1)\\
 & & \text{then}\;(der\,c\,r_1)\cdot r_2 + der\,c\,r_2\\
 & & \text{else}\;(der\,c\,r_1)\cdot r_2\\
der\,c\,(r^*) & \stackrel{\text{def}}{=} & (der\,c\,r)\cdot(r^*)\\
\\
ders\,[]\,r & \stackrel{\text{def}}{=} & r\\
ders\,(c::s)\,r & \stackrel{\text{def}}{=} & ders\,s\,(der\,c\,r)
\end{array}
$$

# Examples

Given $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ what is

$$\begin{aligned} der\, a\, r &= ? \\ der\, b\, r &= ? \\ der\, c\, r &= ? \end{aligned}$$

# The Algorithm

| | | |
|---|---|---|
| Input: | $r_1$, $abc$ | |
| Step 1: | build derivative of $a$ and $r_1$ | $(r_2 = der\,a\,r_1)$ |
| Step 2: | build derivative of $b$ and $r_2$ | $(r_3 = der\,b\,r_2)$ |
| Step 3: | build derivative of $c$ and $r_3$ | $(r_4 = der\,b\,r_3)$ |
| Step 4: | the string is exhausted; test whether $r_4$ can recognise the empty string | $(nullable(r_4))$ |
| Output: | result of the test $\Rightarrow$ *true* or *false* | |

# The Idea of the Algorithm

If we want to recognise the string $abc$ with regular expression $r_1$ then

1. $Der\, a\, (L(r_1))$

# The Idea of the Algorithm

If we want to recognise the string $abc$ with regular expression $r_1$ then

1. $Der\,a\,(L(r_1))$
2. $Der\,b\,(Der\,a\,(L(r_1)))$

# The Idea of the Algorithm

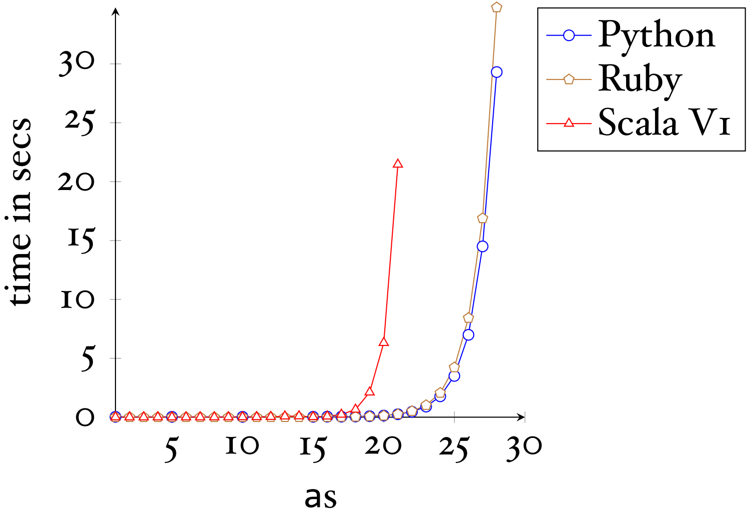
If we want to recognise the string $abc$ with regular expression $r_1$ then

1. $Der\,a\,(L(r_1))$
2. $Der\,b\,(Der\,a\,(L(r_1)))$
3. $Der\,c\,(Der\,b\,(Der\,a\,(L(r_1))))$

4. finally we test whether the empty string is in this set; same for $Ders\,abc\,(L(r_1))$.

The matching algorithm works similarly, just over regular expressions instead of sets.

$$(a^{?\{n\}}) \cdot a^{\{n\}}$$



# A Problem

We represented the "n-times" $a^{\{n\}}$ as a sequence regular expression:

1: $a$

2: $a \cdot a$

3: $a \cdot a \cdot a$

...

13: $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a$

...

20:

This problem is aggravated with $a^?$ being represented as $\epsilon + a$.

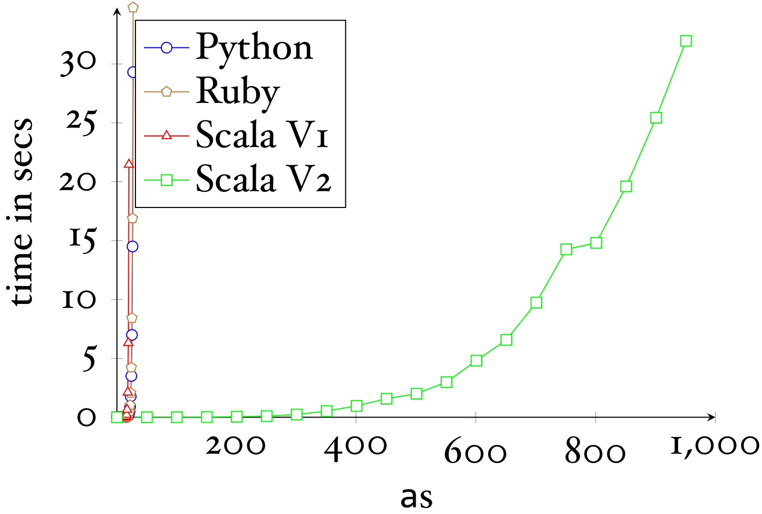
# Solving the Problem

What happens if we extend our regular expressions

$$\begin{array}{rcl} r & ::= & \ldots \\ & | & r^{\{n\}} \\ & | & r^{?} \end{array}$$

What is their meaning? What are the cases for *nullable* and *der*?

$$(a^{?\{n\}}) \cdot a^{\{n\}}$$



Python
Ruby
Scala V1
Scala V2
time in secs
0
5
10
15
20
25
30
200
400
600
800
1,000
as

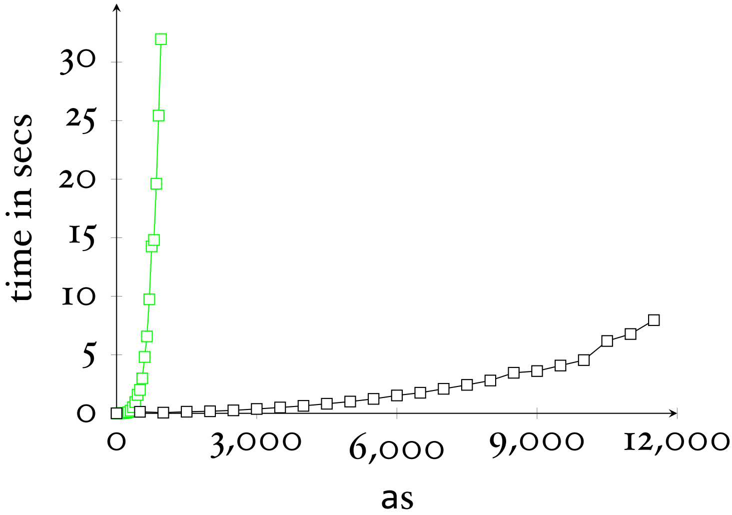
# Examples

Recall the example of $r \stackrel{\text{def}}{=} ((a \cdot b) + b)^*$ with

$$
\begin{aligned}
der\, a\, r &= ((\epsilon \cdot b) + \varnothing) \cdot r \\
der\, b\, r &= ((\varnothing \cdot b) + \epsilon) \cdot r \\
der\, c\, r &= ((\varnothing \cdot b) + \varnothing) \cdot r
\end{aligned}
$$

What are these regular expressions equivalent to?

$$(a^{?\{n\}}) \cdot a^{\{n\}}$$



# What is good about this Alg.

- extends to most regular expressions, for example $\sim r$
- is easy to implement in a functional language
- the algorithm is already quite old; there is still work to be done to use it as a tokenizer (that is brand new work)
- we can prove its correctness...

# Proofs about Rexps

Remember their inductive definition:

$$
\begin{array}{rcl}
r & ::= & \varnothing \\
  & |  & \epsilon \\
  & |  & c \\
  & |  & r_1 \cdot r_2 \\
  & |  & r_1 + r_2 \\
  & |  & r^*
\end{array}
$$

If we want to prove something, say a property $P(r)$, for all regular expressions $r$ then ...

# Proofs about Rexp (2)

- $P$ holds for $\varnothing$, $\epsilon$ and $\mathtt{c}$
- $P$ holds for $r_1 + r_2$ under the assumption that $P$ already holds for $r_1$ and $r_2$.
- $P$ holds for $r_1 \cdot r_2$ under the assumption that $P$ already holds for $r_1$ and $r_2$.
- $P$ holds for $r^*$ under the assumption that $P$ already holds for $r$.

# Proofs about Rexp (3)

Assume $P(r)$ is the property:

$$nullable(r) \text{ if and only if } [] \in L(r)$$

# Proofs about Rexp (4)

$$
\begin{aligned}
rev(\varnothing) &\stackrel{\text{def}}{=} \varnothing \\
rev(\epsilon) &\stackrel{\text{def}}{=} \epsilon \\
rev(c) &\stackrel{\text{def}}{=} c \\
rev(r_1 + r_2) &\stackrel{\text{def}}{=} rev(r_1) + rev(r_2) \\
rev(r_1 \cdot r_2) &\stackrel{\text{def}}{=} rev(r_2) \cdot rev(r_1) \\
rev(r^*) &\stackrel{\text{def}}{=} rev(r)^*
\end{aligned}
$$

We can prove

$$L(rev(r)) = \{s^{-1} \mid s \in L(r)\}$$

by induction on $r$.

# Correctness Proof for our Matcher

- We started from

$$
\begin{aligned}
& s \in L(r) \\
\Leftrightarrow \quad & [] \in Ders\, s\, (L(r))
\end{aligned}
$$

# Correctness Proof for our Matcher

- We started from

$$
\begin{aligned}
& s \in L(r) \\
\Leftrightarrow \quad & [] \in \mathit{Ders}\, s\, (L(r))
\end{aligned}
$$

- if we can show $\mathit{Ders}\, s\, (L(r)) = L(\mathit{ders}\, s\, r)$ we have

$$
\begin{aligned}
\Leftrightarrow \quad & [] \in L(\mathit{ders}\, s\, r) \\
\Leftrightarrow \quad & \mathit{nullable}(\mathit{ders}\, s\, r) \\
\stackrel{\text{def}}{=} \quad & \mathit{matches}\, s\, r
\end{aligned}
$$

# Proofs about Rexp (5)

Let $Der\,c\,A$ be the set defined as

$$Der\,c\,A \stackrel{\text{def}}{=} \{s \mid c::s \in A\}$$

We can prove

$$L(der\,c\,r) = Der\,c\,(L(r))$$

by induction on $r$.

# Proofs about Strings

If we want to prove something, say a property $P(s)$, for all strings $s$ then ...

- $P$ holds for the empty string, and
- $P$ holds for the string $c :: s$ under the assumption that $P$ already holds for $s$

# Proofs about Strings (2)

We can then prove

$$Ders\, s\, (L(r)) = L(ders\, s\, r)$$

We can finally prove

$$matches\, s\, r \text{ if and only if } s \in L(r)$$